Prime Gravity: A Σ_1-Compatible Consistency Certificate

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***Abstract*—We construct a machine–checkable certificate for the analytic number theory underlying the Prime Gravity framework. Working in the setting of analytic number theory, we derive the arithmetic Poisson field and the associated Laplace–Trace identity, then introduce a finite window to obtain an explicit error envelope. All analytic constants are replaced by rational upper bounds, and a finite grid verification protocol is specified. The resulting Σ_1–type certificate can be verified by finite arithmetic on rational numbers and is intended to complement the main Prime Gravity paper by providing a fully formalisable, implementation–independent consistency check of its spectral identities.**

I. Introduction

This document serves as a stand–alone, machine–checkable certificate for the analytic number theory underpinning the *Prime Gravity* framework. Building on the arithmetic Poisson field and the Laplace–Trace identity established in the accompanying research paper, we fix explicit rational bounds for all analytic constants and specify a finite grid verification protocol. The result is a finite Σ_1–type certificate which can be verified by any numerical system without recourse to analytic continuation.

The certificate is intended to complement, rather than replace, the proofs in the main Prime Gravity article. The analytic backbone is provided by the *arithmetic Poisson field* and the *Laplace–Trace identity* proved in the main text. For convenience, we recall these identities here as Proposition II.1 and Theorem II.2 and include short self–contained proofs in Section II; readers willing to take them directly from the main paper may skip those proofs on a first reading. Under these analytic inputs, we construct a finite error envelope for the windowed Laplace–Trace identity and develop a rational ledger and verification protocol expressible in first–order arithmetic.

We summarise the structure of this certificate as follows.

- Section II introduces the formal setting and recalls the principal analytic identities from the main paper.
- Section III develops the windowed error envelope, fixes rational bounds and summarises them in a ledger, together with a list of operational parameters.
- Section IV specifies a finite grid, describes the verification protocol, and formalises the resulting Σ_1 certificate and its verification procedure.
- Appendix A records an optional sample verification log, encoded as a CSV file, illustrating one concrete instance of the finite data described in Definition III.5. This external file is provided only for reproducibility and is not part of the formal mathematical statement of the certificate.

II. Formal Setting and Main Identities

Remark II.1 (Role of this section). The proposition and theorem in this section restate the corresponding results from the main Prime Gravity article [3]. They are included here only to make the present certificate logically self–contained; the finite Σ_1-certificate developed in Sections III and IV uses these analytic identities as input and would remain valid if they were taken as assumptions from the main paper without repeating the proofs.

A. Setting and Notation

Let $\lambda > 0$ be fixed. Let $\Lambda(n)$ denote the von Mangoldt function, and define a signed Radon measure μ on $(0, \infty)$ by

$$\mu := \sum_{n \geq 1} \Lambda(n)\, \delta_{\log n},$$

where δ_x is the unit Dirac mass at $x \in \mathbb{R}$.

Let L_λ be the one–dimensional Yukawa (or shifted Laplacian) operator

$$L_\lambda := -\frac{d^2}{dx^2} + \lambda^2$$

acting on (tempered) distributions on $\mathbb{R}$.

Let $g_\lambda : \mathbb{R} \to \mathbb{R}$ be the Green kernel of L_λ, i.e.

$$g_\lambda(x) := \frac{1}{2\lambda} e^{-\lambda|x|}, \qquad x \in \mathbb{R},$$

so that $g_\lambda \in \mathcal{S}'(\mathbb{R})$ satisfies

$$L_\lambda g_\lambda = \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}).$$

We then define the (distributional) potential U_λ by convolution

$$U_\lambda := g_\lambda * \mu \quad \text{in } \mathcal{D}'(\mathbb{R}),$$

and, when needed, we identify U_λ with its restriction to $(0, \infty)$.

Finally, recall the classical identity

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} = \int_{(0,\infty)} e^{-sx}\, d\mu(x), \qquad \Re s > 1,$$

which expresses the logarithmic derivative of the Riemann zeta function as the Laplace transform of μ.

Notation summary: For the reader's convenience, we collect here the principal symbols used in this certificate.

Symbol	Meaning
$\Lambda(n)$	von Mangoldt function, equal to $\log p$ if $n = p^k$ is a prime power and 0 otherwise.
μ	signed Radon measure $\sum_{n\geq 1} \Lambda(n)\, \delta_{\log n}$ on $(0, \infty)$.
L_λ	Yukawa (shifted Laplacian) operator $-(d^2/dx^2) + \lambda^2$.
g_λ	Green kernel of L_λ, given by $g_\lambda(x) = \dfrac{1}{2\lambda} e^{-\lambda\|x\|}$.
U_λ	distributional potential $g_\lambda * \mu$.
w_ε	scaled even Schwartz window $\varepsilon^{-1} w(u/\varepsilon)$ with $w(u) = \pi^{-1/2} e^{-u^2}$.
$\mathcal{L}_+$	one–sided Laplace transform on $[0, \infty)$, $\mathcal{L}_+\{f\}(s) = \int_0^\infty e^{-sx} f(x)\, dx$.

TABLE I

SUMMARY OF NOTATION USED THROUGHOUT THE CERTIFICATE.

B. Analytic base axioms

Throughout this certificate we fix $\lambda > 0$ and assume the following standard analytic facts from classical number theory and distribution theory.

(AB1) Dirichlet series for $-\zeta'/\zeta$. For $\Re s > 1$ one has the absolutely convergent series

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}.$$

(AB2) Yukawa Green kernel. Let

$$g_\lambda(x) := \frac{1}{2\lambda}\, e^{-\lambda|x|} \qquad (x \in \mathbb{R}).$$

Then $g_\lambda \in \mathcal{D}'_{\exp}(\mathbb{R})$ and, in the sense of distributions,

$$L_\lambda g_\lambda = \delta_0, \qquad L_\lambda := -\frac{d^2}{dx^2} + \lambda^2.$$

(AB3) Growth of the prime source. The measure

$$\mu := \sum_{n\geq 1} \Lambda(n)\, \delta_{\log n}$$

is supported on $(0, \infty)$ and belongs to $\mathcal{D}'_{\exp}(\mathbb{R})$. In particular, for every $\eta > 0$ there exists C_η such that

$$\big|\langle \mu, \varphi\rangle\big| \leq C_\eta \sup_{x\in\mathbb{R}} e^{\eta|x|} |\varphi(x)| \qquad \forall\, \varphi \in \mathcal{D}(\mathbb{R}).$$

(AB4) Laplace transform calculus. If $F \in L^1_{\mathrm{loc}}([0, \infty))$ has at most exponential growth of type $< \Re s$, then the one–sided Laplace transform

$$\mathcal{L}_+\{F\}(s) := \int_0^\infty e^{-sx} F(x)\, dx$$

converges absolutely and depends holomorphically on s in the corresponding half–plane. Moreover, for such F and for $\Re s$ sufficiently large one has

$$\mathcal{L}_+\{F''\}(s) = s^2\, \mathcal{L}_+\{F\}(s)$$

whenever F is twice differentiable with at most exponential growth.

(AB5) Laplace representation of $-\zeta'/\zeta$. For $\Re s > 1$ the measure μ satisfies

$$\mathcal{L}_+\{\mu\}(s) := \int_{(0,\infty)} e^{-sx}\, d\mu(x) = -\frac{\zeta'(s)}{\zeta(s)}.$$

This follows from (AB1) by the change of variables $x = \log n$.

These axioms are the only external analytic input for the certificate. All subsequent propositions and theorems in this document are proved from (AB1)–(AB5) by explicit distributional and Laplace–transform calculations, and the Σ_1–type finite certificate in Section IV depends only on their consequences.

C. Arithmetic Poisson Field

Proposition II.1 (Arithmetic Poisson Field)**.** *With the notation above, let $U_\lambda := g_\lambda * \mu$ in $\mathcal{D}'_{\exp}(\mathbb{R})$. Then*

$$L_\lambda U_\lambda = \mu \qquad \text{in } \mathcal{D}'(\mathbb{R}),$$

with $L_\lambda = -\frac{d^2}{dx^2} + \lambda^2$ and μ supported on $(0, \infty)$. Moreover U_λ has at most exponential growth of type λ, and $U_\lambda|_{(0,\infty)}$ is locally integrable.

Proof. By (AB2) the Yukawa kernel g_λ belongs to $\mathcal{D}'_{\exp}(\mathbb{R})$ and satisfies $L_\lambda g_\lambda = \delta_0$ in $\mathcal{D}'(\mathbb{R})$. By (AB3) the source measure μ is also in $\mathcal{D}'_{\exp}(\mathbb{R})$, so the convolution

$$U_\lambda := g_\lambda * \mu$$

is well–defined in $\mathcal{D}'_{\exp}(\mathbb{R})$ and has at most exponential growth of type λ.

Let $\varphi \in \mathcal{D}(\mathbb{R})$ be arbitrary. Using the usual convolution adjoint relation for distributions, one has

$$\langle L_\lambda U_\lambda, \varphi\rangle = \langle U_\lambda, L_\lambda\varphi\rangle = \langle g_\lambda * \mu, L_\lambda\varphi\rangle = \langle \mu, \tilde{g}_\lambda * L_\lambda\varphi\rangle,$$

where $\tilde{g}_\lambda(x) := g_\lambda(-x)$. By (AB2), $L_\lambda g_\lambda = \delta_0$ and the formal self–adjointness of L_λ imply

$$\tilde{g}_\lambda * L_\lambda\varphi = \varphi \qquad \text{in } \mathcal{D}(\mathbb{R}),$$

so the last pairing equals $\langle \mu, \varphi\rangle$. Thus

$$\langle L_\lambda U_\lambda, \varphi\rangle = \langle \mu, \varphi\rangle \qquad \forall\, \varphi \in \mathcal{D}(\mathbb{R}),$$

which is precisely $L_\lambda U_\lambda = \mu$ in $\mathcal{D}'(\mathbb{R})$.

The exponential growth and local integrability of U_λ on $(0, \infty)$ follow from the exponential bounds in (AB2) and (AB3) by standard estimates on convolutions of exponentially bounded distributions, and we omit the routine details. ☐

D. Laplace–Trace identity

Theorem II.2 (Laplace–Trace identity)**.** *For $\Re s > 1$ one has*

$$(s^2 - \lambda^2)\, \mathcal{L}_+\{U_\lambda\}(s) = -\frac{\zeta'(s)}{\zeta(s)}.$$

Equivalently,

$$(s^2 - \lambda^2) \int_0^\infty e^{-sx} U_\lambda(x)\, dx = -\frac{\zeta'(s)}{\zeta(s)} \qquad (\Re s > 1).$$

Proof. Let $s \in \mathbb{C}$ with $\Re s > 1$. By Proposition II.1 the distribution U_λ satisfies $L_\lambda U_\lambda = \mu$ in $\mathcal{D}'(\mathbb{R})$ and has at most exponential growth of type λ. Restricting to $x > 0$ and applying (AB4), we see that $\mathcal{L}_+\{U_\lambda\}(s)$ converges absolutely and depends holomorphically on s in a right half–plane containing $\Re s > 1$, and that

$$\mathcal{L}_+\{L_\lambda U_\lambda\}(s) = (s^2 - \lambda^2)\,\mathcal{L}_+\{U_\lambda\}(s) \qquad (\Re s > 1).$$

On the other hand, (AB5) gives

$$\mathcal{L}_+\{\mu\}(s) = -\frac{\zeta'(s)}{\zeta(s)} \qquad (\Re s > 1).$$

Since $L_\lambda U_\lambda = \mu$ in $\mathcal{D}'(\mathbb{R})$, taking the Laplace transform on both sides yields

$$\mathcal{L}_+\{L_\lambda U_\lambda\}(s) = \mathcal{L}_+\{\mu\}(s) = -\frac{\zeta'(s)}{\zeta(s)}.$$

Combining the two expressions for $\mathcal{L}_+\{L_\lambda U_\lambda\}(s)$ proves the stated identity. □

E. Archimedean closure and completed trace

Define the completed zeta and its logarithmic derivative by

$$\xi(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \Xi(s) := -\frac{\xi'(s)}{\xi(s)}.$$

Theorem II.3 (Archimedean closure). *There exists a distribution $\mu_\infty \in \mathcal{D}'_{\exp}([0,\infty))$ such that for $\Re s > 1$*

$$\mathcal{L}_+\{\mu_\infty\}(s) = \frac{1}{2}\log\pi - \frac{1}{2}\psi\Big(\frac{s}{2}\Big),$$

where ψ denotes the digamma function. Let

$$\widetilde{\mu} := \mu + \mu_\infty, \qquad \widetilde{U}_\lambda := g_\lambda * \widetilde{\mu}.$$

Then for $\Re s > 1$ one has the completed trace identity

$$(s^2 - \lambda^2)\,\mathcal{L}_+\{\widetilde{U}_\lambda\}(s) = \Xi(s).$$

Proof. The analytic factor

$$\log\Gamma\Big(\frac{s}{2}\Big) - \frac{s}{2}\log\pi$$

admits a classical Laplace representation on $\Re s > 1$; differentiating with respect to s yields

$$\frac{1}{2}\log\pi - \frac{1}{2}\psi\Big(\frac{s}{2}\Big) = \mathcal{L}_+\{\mu_\infty\}(s)$$

for some $\mu_\infty \in \mathcal{D}'_{\exp}([0,\infty))$ with at most exponential growth. (An explicit formula for μ_∞ can be written in terms of a rapidly decaying kernel and we do not need its exact shape here.)

By definition,

$$\Xi(s) = -\frac{\xi'(s)}{\xi(s)} = -\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2}\log\pi - \frac{1}{2}\psi\Big(\frac{s}{2}\Big).$$

Using (AB5) and the definition of μ_∞, this can be rewritten as

$$\Xi(s) = \mathcal{L}_+\{\mu\}(s) + \mathcal{L}_+\{\mu_\infty\}(s) = \mathcal{L}_+\{\widetilde{\mu}\}(s) \qquad (\Re s > 1).$$

Finally, by the same argument as in the proof of Theorem II.2, applied to $\widetilde{\mu}$ in place of μ, we obtain

$$(s^2 - \lambda^2)\,\mathcal{L}_+\{\widetilde{U}_\lambda\}(s) = \mathcal{L}_+\{\widetilde{\mu}\}(s) = \Xi(s),$$

which is the claimed completed identity. □

III. Error Envelope and Rational Ledger

Remark III.1 (Two–layer structure of the certificate). All analytic identities used in this section and in Section IV are consequences of the base axioms (AB1)–(AB5) stated in Section II-B via Proposition II.1, Theorem II.2 and, for the completed factor, Theorem II.3. The role of Sections III–IV is purely Σ_1–type: they replace all analytic constants by explicitly given rational bounds and specify a finite grid and verification protocol whose outcome is a Boolean pattern reproducible on any platform.

In order to obtain a finite and explicitly controllable error envelope, we introduce an even Schwartz window on the physical side and track its effect as a multiplier on the spectral side.

A. Window and Multiplier

Let

$$w(u) := \pi^{-1/2}e^{-u^2}, \qquad u \in \mathbb{R},$$

and for $\varepsilon > 0$ define the scaled Gaussian

$$w_\varepsilon(u) := \varepsilon^{-1}w(u/\varepsilon) = \frac{1}{\varepsilon\sqrt{\pi}}\,e^{-(u/\varepsilon)^2}.$$

Its bilateral Laplace transform is the classical Gaussian factor

$$M_\varepsilon(s) := \int_{-\infty}^{\infty} e^{-su}w_\varepsilon(u)\,du = \exp\Big(\frac{\varepsilon^2 s^2}{4}\Big), \qquad s \in \mathbb{C}.$$

In this certificate we use the Gaussian as a *spectral window* and define the windowed potential $U_{\lambda,\varepsilon}$ directly on the Laplace side.

Definition III.1 (Windowed potential). *For each $\varepsilon > 0$ we define the (tempered) distribution $U_{\lambda,\varepsilon}$ on $(0,\infty)$ to be the unique object whose one–sided Laplace transform satisfies*

$$\mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) := M_\varepsilon(s)\,\mathcal{L}_+\{U_\lambda\}(s), \qquad \Re s > 1.$$

By standard properties of the Laplace transform, this prescription determines $U_{\lambda,\varepsilon}$ uniquely up to equality in $\mathcal{D}'(0,\infty)$. Intuitively, $U_{\lambda,\varepsilon}$ is obtained from U_λ by applying a Gaussian window in the spectral variable.

B. Pointwise Bound for the Multiplier

We record a simple but crucial bound for $M_\varepsilon(s) - 1$.

Lemma III.2 (Exponential bound for the multiplier). *For every $\varepsilon > 0$ and $s \in \mathbb{C}$, one has*

$$\big|M_\varepsilon(s) - 1\big| = \Big|\exp\Big(\frac{\varepsilon^2 s^2}{4}\Big) - 1\Big| \le \exp\Big(\frac{\varepsilon^2|s|^2}{4}\Big) - 1.$$

Proof. Let $z := \varepsilon^2 s^2/4$. Then $M_\varepsilon(s) = e^z$ and

$$|e^z - 1| \le e^{|z|} - 1$$

for all $z \in \mathbb{C}$, by the triangle inequality applied to the power series of the exponential. Since $|z| = \varepsilon^2|s|^2/4$, the desired inequality follows. □

C. Tail Bound for the Logarithmic Derivative

We now isolate a uniform bound for the logarithmic derivative of the Riemann zeta function on a right half-plane.

Lemma III.3 (Dirichlet tail bound)**.** *Let $\delta > 0$ be fixed. Then there exists a finite constant $B_\delta > 0$ such that*

$$\left|-\frac{\zeta'(s)}{\zeta(s)}\right| \le B_\delta \qquad \text{for all } s \text{ with } \Re s \ge 1+\delta.$$

Proof. For $\Re s > 1$ one has the absolutely convergent Dirichlet series

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\ge 1} \frac{\Lambda(n)}{n^s}.$$

Fix $\delta > 0$ and write $s = \sigma + it$ with $\sigma \ge 1 + \delta$. Then

$$\left|-\frac{\zeta'(s)}{\zeta(s)}\right| \le \sum_{n\ge 1} \frac{\Lambda(n)}{n^\sigma} \le \sum_{n\ge 1} \frac{\Lambda(n)}{n^{1+\delta}} =: B_\delta,$$

and $B_\delta < \infty$ by comparison with a convergent integral. This yields the claimed uniform bound on the half-plane $\Re s \ge 1+\delta$. □

D. Error Envelope for the Windowed Trace

We now combine the previous lemmas to obtain an explicit error envelope for the difference between the original and windowed Laplace–Trace identities.

For s with $\Re s > 1$ we set

$$E(s) := (s^2 - \lambda^2)\,\mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) + \frac{\zeta'(s)}{\zeta(s)}.$$

Proposition III.4 (Error envelope)**.** *Fix $\delta > 0$, and choose parameters $\varepsilon_0 > 0$ and $S_0 > 0$. Then there exists a constant $C_\delta > 0$, depending only on δ, ε_0 and S_0, such that for all*

$$0 < \varepsilon \le \varepsilon_0, \qquad \Re s \ge 1+\delta, \qquad |s| \le S_0,$$

one has

$$|E(s)| \le C_\delta\, \varepsilon^2 |s|^2.$$

Proof. By Definition III.1 and Theorem II.2, we have for $\Re s > 1$

$$(s^2-\lambda^2)\,\mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) = (s^2-\lambda^2)\,M_\varepsilon(s)\,\mathcal{L}_+\{U_\lambda\}(s) = M_\varepsilon(s)\left(-\frac{\zeta'(s)}{\zeta(s)}\right).$$

Thus

$$E(s) = \Big(M_\varepsilon(s) - 1\Big)\left(-\frac{\zeta'(s)}{\zeta(s)}\right),$$

and therefore

$$|E(s)| \le \big|M_\varepsilon(s) - 1\big| \left|-\frac{\zeta'(s)}{\zeta(s)}\right|.$$

On the one hand, Lemma III.2 gives

$$\big|M_\varepsilon(s) - 1\big| \le \exp\!\Big(\frac{\varepsilon^2|s|^2}{4}\Big) - 1.$$

On the other hand, Lemma III.3 yields

$$\left|-\frac{\zeta'(s)}{\zeta(s)}\right| \le B_\delta \qquad (\Re s \ge 1+\delta).$$

Now restrict to $0 < \varepsilon \le \varepsilon_0$ and $|s| \le S_0$. The function

$$\Phi(u) := \frac{e^u - 1}{u}, \qquad u > 0,$$

is continuous and increasing on $(0,\infty)$, hence

$$\exp\!\Big(\frac{\varepsilon^2|s|^2}{4}\Big) - 1 = \Phi\Big(\frac{\varepsilon^2|s|^2}{4}\Big)\,\frac{\varepsilon^2|s|^2}{4} \le \Phi\Big(\frac{\varepsilon_0^2 S_0^2}{4}\Big)\,\frac{\varepsilon^2|s|^2}{4}.$$

Combining the inequalities, we obtain

$$|E(s)| \le B_\delta\,\Phi\Big(\frac{\varepsilon_0^2 S_0^2}{4}\Big)\,\frac{\varepsilon^2|s|^2}{4}.$$

Setting

$$C_\delta := \frac{B_\delta}{4}\,\Phi\Big(\frac{\varepsilon_0^2 S_0^2}{4}\Big)$$

gives the desired bound

$$|E(s)| \le C_\delta\,\varepsilon^2|s|^2$$

for all admissible ε and s. □

E. Σ_1-type Finite Certificate

In order to obtain a verifiable certificate in the sense of first–order arithmetic, we now fix *rational* upper bounds for all analytic constants that appear in the error envelope of Proposition III.4. All bounds are chosen with *outward rounding*, so that any replacement of a real quantity by its rational bound makes the inequalities strictly safer.

1) Rational bounds for analytic constants: We first fix a rational upper bound for π:

$$\pi_{\mathrm{up}} := \frac{355}{113},$$

which satisfies $\pi < \pi_{\mathrm{up}}$ and is classically known.

Next, for a fixed $\delta > 0$, we define the Dirichlet tail constant

$$B_\delta := \sum_{n\ge 1} \frac{\Lambda(n)}{n^{1+\delta}},$$

which is finite by Lemma III.3. We choose a truncation index $N_\delta \in \mathbb{N}$ and set

$$B_\delta^{\mathrm{cap}} := \sum_{n=1}^{N_\delta} \frac{\Lambda(n)}{n^{1+\delta}} \;+\; T_\delta^{\mathrm{up}},$$

where $T_\delta^{\mathrm{up}} \in \mathbb{Q}_{>0}$ is any rational number satisfying

$$T_\delta^{\mathrm{up}} \;\ge\; \sum_{n > N_\delta} \frac{\Lambda(n)}{n^{1+\delta}}.$$

By construction $B_\delta \le B_\delta^{\mathrm{cap}}$, and B_δ^{cap} is a rational number.

We also fix rational parameters

$$\varepsilon_0 \in \mathbb{Q}_{>0}, \qquad S_0 \in \mathbb{Q}_{>0},$$

and define the maximal argument for the exponential multiplier by

$$U_{\max} := \frac{\varepsilon_0^2 S_0^2}{4} \in \mathbb{Q}_{>0}.$$

To bound the function

$$u \longmapsto e^u - 1, \qquad 0 \le u \le U_{\max},$$

Quantity	Rational bound	Method
π	$\pi_{\mathrm{up}} = \dfrac{355}{113}$	Classical rational
upper bound		
B_δ	B_δ^{cap}	Dirichlet sum up to N_δ
+ rational tail bound		
$e^u - 1$ on $[0, U_{\max}]$	$R_{\exp}^{\mathrm{up}}$	Truncated series
+ rational tail bound		
C_δ in Prop. III.4	C_δ^{cap}	Product of B_δ^{cap}
and $R_{\exp}^{\mathrm{up}}/U_{\max}$		

we choose a truncation order $K \in \mathbb{N}$ and set

$$R_{\exp}^{\mathrm{up}} := \sum_{k=1}^{K} \frac{U_{\max}^k}{k!} \; + \; R_{\mathrm{tail}}^{\mathrm{up}},$$

where $R_{\mathrm{tail}}^{\mathrm{up}} \in \mathbb{Q}_{>0}$ is a rational upper bound for the remaining tail $\sum_{k>K} U_{\max}^k/k!$. Then

$$e^u - 1 \; \leq \; R_{\exp}^{\mathrm{up}} \qquad \text{for all } 0 \leq u \leq U_{\max},$$

and $R_{\exp}^{\mathrm{up}} \in \mathbb{Q}_{>0}$ is again rational.

Putting these pieces together, we obtain a rational upper bound for the constant C_δ in Proposition III.4. Indeed, recalling

$$C_\delta = \frac{B_\delta}{4}\, \Phi\Big(\frac{\varepsilon_0^2 S_0^2}{4}\Big), \qquad \Phi(u) = \frac{e^u - 1}{u},$$

we may set

$$C_\delta^{\mathrm{cap}} := \frac{B_\delta^{\mathrm{cap}}}{4}\, \frac{R_{\exp}^{\mathrm{up}}}{U_{\max}} \in \mathbb{Q}_{>0},$$

and then

$$C_\delta \; \leq \; C_\delta^{\mathrm{cap}}.$$

2) Rational-bound ledger: For later use, it is convenient to summarise the above choices in a finite table (the "ledger" of the certificate):

By construction, every entry in the "Rational bound" column is a positive rational number, and each is an *outer* (safe) bound for the corresponding analytic quantity. In particular, any occurrence of π, B_δ, $e^u - 1$ and C_δ in the error estimates of Proposition III.4 can be replaced by these rational bounds without invalidating any inequality.

3) Parameter choices: To operationalise the certificate one must fix positive rational parameters that control the domain of verification and the truncation indices in the rational ledger. Table II lists the parameters appearing in the certificate and their roles.

Parameter	Description
δ_0	Lower bound shift for $\Re s$ in the verification domain.
ε_0	Maximum window width: $0 < \varepsilon \leq \varepsilon_0$.
S_0	Bound for the modulus $\lvert s \rvert$ in the verification domain.
N_δ	Truncation index for the Dirichlet tail in B_δ^{cap}.
K	Truncation order for the exponential series in $R_{\exp}^{\mathrm{up}}$.

TABLE II
OPERATIONAL PARAMETERS USED IN THE FINITE Σ_1 CERTIFICATE.

Example instantiation of parameters: For the sample verification log in Appendix A, we fix one concrete set of rational parameters as follows:

$$\delta_0 := \frac{1}{10}, \qquad \varepsilon_0 := \frac{1}{100}, \qquad S_0 := 25,$$
$$N_{\delta_0} := 10^7, \qquad K := 20.$$

Using these choices we compute explicit rational values

$$B_{\delta_0}^{\mathrm{cap}}, \quad R_{\exp}^{\mathrm{up}}, \quad C_{\delta_0}^{\mathrm{cap}} \in \mathbb{Q}_{>0},$$

by the formulas in the rational ledger (Dirichlet truncation plus a rational tail bound for B_{δ_0}, truncated exponential series plus rational tail for $R_{\exp}^{\mathrm{up}}$, and the product defining $C_{\delta_0}^{\mathrm{cap}}$). For the concrete numerical run used in the sample verification log of Appendix A, their decimal expansions are encoded in a CSV file. This encoding is merely a convenient representation of the finite rational data; the logical content of the certificate depends only on the existence of such rationals satisfying the stated inequalities. In the text of this certificate we use the symbols $B_{\delta_0}^{\mathrm{cap}}$, $R_{\exp}^{\mathrm{up}}$, and $C_{\delta_0}^{\mathrm{cap}}$ as fixed rational constants.

4) Σ_1 character of the certificate: We can now summarize the logical form of the error control.

Definition III.5 (Finite Σ_1-certificate)**.** *Fix $\delta > 0$, and choose rational parameters*

$$\varepsilon_0,\, S_0,\, \pi_{\mathrm{up}},\, B_\delta^{\mathrm{cap}},\, R_{\exp}^{\mathrm{up}},\, C_\delta^{\mathrm{cap}} \in \mathbb{Q}_{>0}$$

as above. A finite Σ_1-certificate *for the windowed Laplace–Trace identity on the domain*

$$0 < \varepsilon \leq \varepsilon_0, \qquad \Re s \geq 1 + \delta, \qquad |s| \leq S_0,$$

is the finite data consisting of:

- *the rational-bound ledger in the table above;*
- *a finite grid $\mathcal{G} \subset \{s \in \mathbb{C} : \Re s \geq 1 + \delta,\ |s| \leq S_0\}$;*
- *for each $s \in \mathcal{G}$, a verification that the rational inequality*

  $$|E(s)| \; \leq \; C_\delta^{\mathrm{cap}}\, \varepsilon^2 |s|^2$$

  holds when all analytic quantities in $E(s)$ are evaluated with outward rounding and all constants are replaced by their rational bounds from the ledger.

All objects appearing in a finite Σ_1-certificate are therefore:

- natural numbers (indices, truncation orders, grid sizes),
- rational numbers (bounds and evaluated values),
- and finitely many inequalities between rationals.

In particular, the statement "there exists a finite Σ_1-certificate with these parameters" has purely existential form and can be expressed as a Σ_1-formula in first–order arithmetic.

IV. Grid-based Verification and Σ_1 Procedure

The purpose of this subsection is to specify an explicit, finite grid in the s–plane and to record a verification protocol which can be implemented by any numerical system using only outward rounding and rational bounds, as described in Definition III.5.

A. Choice of grid

As a concrete example, we fix

$$\delta_0 := 0.1, \qquad \varepsilon_0 > 0, \qquad S_0 > 0$$

with $S_0 \geq 25$, and we consider the rectangular domain

$$\mathcal{D} := \{s = \sigma + it \in \mathbb{C} : \sigma \geq 1+\delta_0,\ |t| \leq 20\} \subset \{\Re s \geq 1+\delta_0,\ |s| \leq S_0\}.$$

We then discretise $\mathcal{D}$ with the finite grid

$$\mathcal{G} := \{s_{ij} = \sigma_i + it_j : i = 0, 1, \dots, 30,\ j = 0, 1, \dots, 40\},$$

where

$$\sigma_i := 1.1 + 0.01\, i, \qquad t_j := -20 + j.$$

By construction, $\Re s_{ij} = \sigma_i \geq 1.1 = 1 + \delta_0$ and $|t_j| \leq 20$, so that $\mathcal{G} \subset \mathcal{D}$.

B. Quantities to be evaluated on the grid

Recall the error term

$$E(s) := (s^2 - \lambda^2)\, \mathcal{L}_+\{U_{\lambda,\varepsilon}\}(s) + \frac{\zeta'(s)}{\zeta(s)},$$

and the rational upper bound $C_{\delta_0}^{\mathrm{cap}}$ from the ledger in the previous subsection. For each grid point $s_{ij} \in \mathcal{G}$ we define the rational quantity

$$R(s_{ij}) := C_{\delta_0}^{\mathrm{cap}}\, \varepsilon^2\, |s_{ij}|^2.$$

In practice, every occurrence of $|E(s_{ij})|$ and $|s_{ij}|^2$ will be replaced by rational approximations computed with outward rounding:

- $|E(s_{ij})|_{\mathrm{up}} \in \mathbb{Q}_{>0}$ is a rational upper bound for $|E(s_{ij})|$ obtained by evaluating all sums and products in the definition of $E(s_{ij})$ with upward rounding and by replacing all analytic constants by their rational bounds from the ledger;
- $R(s_{ij})_{\mathrm{down}} \in \mathbb{Q}_{>0}$ is a rational lower bound for $R(s_{ij})$, obtained by computing $|s_{ij}|^2$ with downward rounding and then multiplying by $C_{\delta_0}^{\mathrm{cap}}\varepsilon^2$ with downward rounding.

1) Dirichlet-series evaluation of $-\zeta'/\zeta$: For each grid point $s_{ij} \in \mathcal{G}$ we evaluate

$$-\frac{\zeta'(s_{ij})}{\zeta(s_{ij})}$$

by a finite Dirichlet series together with a rational tail bound derived from $B_{\delta_0}^{\mathrm{cap}}$.

Let $s_{ij} = \sigma_i + it_j$ with $\sigma_i \geq 1+\delta_0$ as in the grid definition, and let $N_* := N_{\delta_0}$ be the truncation index from the rational ledger. We perform the following steps:

1) **Finite Dirichlet sum.**
Compute the partial sum

$$S_{N_*}(s_{ij}) := \sum_{n=1}^{N_*} \frac{\Lambda(n)}{n^{s_{ij}}},$$

using finite arithmetic on rationals with outward rounding; the real and imaginary parts of $S_{N_*}(s_{ij})$ are stored as rationals.

2) **Uniform tail bound.**
Using $\sigma_i \geq 1 + \delta_0$ and Lemma III.3, we have

$$\left|\sum_{n>N_*} \frac{\Lambda(n)}{n^{s_{ij}}}\right| \leq \sum_{n>N_*} \frac{\Lambda(n)}{n^{\sigma_i}} \leq \sum_{n>N_*} \frac{\Lambda(n)}{n^{1+\delta_0}}.$$

By the definition of $B_{\delta_0}^{\mathrm{cap}}$ we set

$$T_{\delta_0}^{\mathrm{up}} := B_{\delta_0}^{\mathrm{cap}} - \sum_{n=1}^{N_*} \frac{\Lambda(n)}{n^{1+\delta_0}},$$

where the finite sum on the right-hand side is evaluated with downward rounding. Then $T_{\delta_0}^{\mathrm{up}} \in \mathbb{Q}_{>0}$ is a rational upper bound for $\sum_{n>N_*} \Lambda(n)/n^{1+\delta_0}$, and in particular

$$\left|\sum_{n>N_*} \frac{\Lambda(n)}{n^{s_{ij}}}\right| \leq T_{\delta_0}^{\mathrm{up}}.$$

3) **Rational enclosure.**
We define a rational upper bound for $\left|-\zeta'(s_{ij})/\zeta(s_{ij})\right|$ by

$$B(s_{ij})_{\mathrm{up}} := \left|S_{N_*}(s_{ij})\right|_{\mathrm{up}} + T_{\delta_0}^{\mathrm{up}},$$

where $\left|S_{N_*}(s_{ij})\right|_{\mathrm{up}}$ is obtained by computing the modulus of $S_{N_*}(s_{ij})$ with outward rounding on rationals. Then

$$\left|-\frac{\zeta'(s_{ij})}{\zeta(s_{ij})}\right| \leq B(s_{ij})_{\mathrm{up}}.$$

In the subsequent error estimates we may replace $\left|-\zeta'(s_{ij})/\zeta(s_{ij})\right|$ by the rational quantity $B(s_{ij})_{\mathrm{up}}$ without weakening any inequality.

C. Verification protocol

The grid-based verification protocol is then the following finite procedure:

1) Fix rational parameters

$$\delta_0 = 0.1, \quad \varepsilon \in (0, \varepsilon_0], \quad \varepsilon_0,\ S_0,\ \pi_{\mathrm{up}},\ B_{\delta_0}^{\mathrm{cap}},\ R_{\mathrm{exp}}^{\mathrm{up}},\ C_{\delta_0}^{\mathrm{cap}}$$

as in the rational-bound ledger.

2) Construct the grid $\mathcal{G} = \{s_{ij}\}$ with

$$\sigma_i = 1.1 + 0.01\, i\ (i = 0, \dots, 30), \qquad t_j = -20 + j\ (j = 0, \dots, 40).$$

3) For each pair (i, j):
   a) evaluate $E(s_{ij})$ numerically using outward rounding and rational bounds for all analytic constants, and record a rational upper bound $|E(s_{ij})|_{\mathrm{up}}$;
   b) evaluate $R(s_{ij})$ with downward rounding to obtain a rational lower bound $R(s_{ij})_{\mathrm{down}}$;
   c) check the rational inequality

$$|E(s_{ij})|_{\mathrm{up}} \ \leq\ R(s_{ij})_{\mathrm{down}}.$$

4) Record, in a finite table (the verification log), the triple

$$\left(s_{ij},\ |E(s_{ij})|_{\mathrm{up}},\ R(s_{ij})_{\mathrm{down}}\right)$$

together with the Boolean result of the comparison for each (i, j).

Definition IV.1 (Grid-based verification log)**.** *A* grid-based verification log *for the windowed Laplace–Trace identity on $\mathcal{D}$ consists of:*

- *the rational-bound ledger of all constants;*
- *the explicit description of the grid $\mathcal{G}$ as above;*
- *for each $s_{ij} \in \mathcal{G}$, rational numbers $|E(s_{ij})|_{\mathrm{up}}$, $R(s_{ij})_{\mathrm{down}}$ and the verification that*

$$|E(s_{ij})|_{\mathrm{up}} \;\le\; R(s_{ij})_{\mathrm{down}}.$$

If all these inequalities hold, we say that the grid-based verification log supports *the finite Σ_1-certificate of Definition III.5 on the domain $\mathcal{D}$.*

D. Σ_1 verification procedure

We finally make explicit the logical and computational constraints which ensure that the finite certificate is verifiable by any numerical system, without ambiguity or dependence on analytic continuation.

Definition IV.2 (Σ_1 verification procedure)**.** *A verification procedure for the windowed Laplace–Trace identity on a domain $\mathcal{D}$ (such as the grid domain of Definition IV.1) is said to be a Σ_1* verification procedure *if it satisfies the following conditions:*

1) ***All constants are rational.***
Every constant appearing in the computation (including bounds for π, B_δ, $e^u - 1$, C_δ, grid coordinates, truncation indices, etc.) is represented by a rational number in $\mathbb{Q}$.

2) ***All inequalities use outward rounding.***
Whenever a real quantity is evaluated numerically, it is replaced by a rational upper or lower bound obtained by outward rounding, so that every inequality is checked in a strictly safe direction.

3) ***All computations are finite.***
The procedure uses only:
   - *finite sums;*
   - *finite products;*
   - *evaluation of the error term $E(s)$ on a finite grid $\mathcal{G} \subset \mathcal{D}$, together with the comparison of finitely many rational inequalities.*

   In particular, no limiting process and no infinite series are evaluated symbolically at run time.

4) ***No analytic continuation is invoked.***
All occurrences of ζ, ζ'/ζ, and related analytic objects are evaluated only in regions where their defining Dirichlet series or integrals converge absolutely. Analytic continuation is never used as a black box in the certificate.

5) ***Software–independent Boolean outcome.***
*Since the procedure consists solely of finitely many arithmetic operations on rational numbers with prescribed outward rounding, any implementation on any software or hardware platform produces the same list of Boolean outcomes (*True/False*) for all grid inequalities.*

Under these constraints, the statement

$$\exists \text{ finite ledger, grid and verification log satisfying all inequalities}$$

is a purely existential statement about natural numbers and rational numbers in first–order arithmetic. Any concrete file format (for example, a CSV file storing the rows of the verification log) is simply a human–readable encoding of such a finite witness and is not part of the formal statement itself. In particular, if one system produces a successful verification log, then any other system following the same Σ_1 verification procedure will reproduce the same True/False pattern.

Appendix

For convenience, we provide one sample grid-based verification log, encoded as an external CSV file. This file is not part of the formal mathematical statement of the certificate; it is merely a concrete realisation of the finite verification log appearing in Definition IV.1 for the particular parameter choices described in Section III-E3. The CSV file contains the grid points s_{ij} together with the rational bounds $|E(s_{ij})|_{\mathrm{up}}$ and $R(s_{ij})_{\mathrm{down}}$, and the Boolean outcome of each comparison. Researchers and verifiers may either recompute an equivalent verification log from the specification in Section IV or consult this CSV file as a reference implementation.

References

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